Introduction to Mathematics and Modeling

lecture 1
First order differential equations
This week

1. Section 9.1: solutions, slope fields, Euler’s method
2. Section 9.2: first-order linear equations
3. Section 9.3: applications
What is a differential equation?

**Definition**

- **A (first order) differential equation** is an equation involving an unknown function and its derivatives

\[ F(x, y, y') = 0 \]

- A **solution** is a function \( y(x) \), that satisfies the differential equation:

\[ F\left(x, y(x), \frac{dy(x)}{dx}\right) = 0. \]

**Definition**

A **normal (first order) differential equation** is an equation of the form

\[ y' = f(x, y) \]
What is a differential equation?

Example:

\[ \frac{dy}{dx} = \cos(x) \]

The function \( y(x) = \sin(x) \) is a solution because

\[ \frac{dy}{dx} = \frac{d}{dx}(\sin(x)) = \cos(x) \]

Every anti-derivative of \( \cos(x) \) is a solution of (*).

The solutions of (*) are \( y(x) = \sin(x) + C \) with \( C \) an arbitrary constant.
What is a differential equation?

Example:

\[ y' = \cos(x) \]  (*)

The function \( y(x) = \sin(x) \) is a solution because

\[
\frac{d}{dx} y(x) = \frac{d}{dx} (\sin(x)) = \cos(x).
\]
What is a differential equation?

Example:

\[ y' = \cos(x) \]  

\[ (\ast) \]

- The function \( y(x) = \sin(x) \) is a solution because

\[ \frac{d y(x)}{d x} = \frac{d}{d x} (\sin(x)) = \cos(x). \]

- Every anti-derivative of \( \cos(x) \) is a solution of \((\ast)\).
What is a differential equation?

Example:

\[ y' = \cos(x) \]  

\( (*) \)

- The function \( y(x) = \sin(x) \) is a solution because
  \[
  \frac{d}{dx} y(x) = \frac{d}{dx} (\sin(x)) = \cos(x).
  \]

- Every anti-derivative of \( \cos(x) \) is a solution of \( (*) \).

- The solutions of \( (*) \) are
  \[
  y(x) = \sin(x) + C
  \]
  with \( C \) an arbitrary constant.
Example:

\[ y' = 2xy \]

The function \( y(x) = e^{x^2} \) is a solution because

\[ \frac{dy(x)}{dx} = 2x e^{x^2} = 2x y(x). \]
What is a differential equation?

Example:

\[ y' = 2xy \]

- The function \( y(x) = e^{x^2} \) is a solution because

\[
\frac{dy(x)}{dx} = 2x e^{x^2} = 2x y(x).
\]

- For every \( C \) the function \( y(x) = Ce^{x^2} \) is a solution.
What is a differential equation?

\[ y' = f(x, y) \]

- It is easy to check whether given function \( y(x) \) is a solution.
What is a differential equation?

\[ y' = f(x, y) \]

- It is easy to check whether given function \( y(x) \) is a solution.
- Solutions of differential equations contain an arbitrary constant \( C \).

Definition

An additional condition like \( y(x_0) = y_0 \) where \( x_0 \) and \( y_0 \) are given values is called an initial condition or boundary condition.
What is a differential equation?

\[
y' = f(x, y)
\]

- It is easy to check whether given function \(y(x)\) is a solution.
- Solutions of differential equations contain an arbitrary constant \(C\).
- The constant \(C\) can be determined by specifying \(y(0)\).
What is a differential equation?

\[ y' = f(x, y) \]

- It is easy to check whether given function \( y(x) \) is a solution.
- Solutions of differential equations contain an arbitrary constant \( C \).
- The constant \( C \) can be determined by specifying \( y(0) \).

**Definition**

**An additional condition like** \( y(x_0) = y_0 \) **where** \( x_0 \) **and** \( y_0 \) **are given values is called an initial condition or boundary condition.**
Slope fields

$y(x)$ is solution of $y' = f(x, y)$ passing through $y_0 = y(x_0)$

The slope of $\ell$ is $y'(x_0) = f(x_0, y_0)$
$y(x_0) = y_0$

$y(x)$ is solution of $y' = f(x, y)$ passing through $y_0 = y(x_0)$

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Slope fields

\[ y' = 0 \]

\[ y' = 0 \]
$y' = 0$

$y(x) = C'$
Slope fields

\[ y' = y \]
\[ y' = y \]

\[ y(x) = Ce^x \]
Slope fields

\[ y' = y - x \]

\[ y(x) = 1 + x - Ce^x \]
y' = y - x

y(x) = 1 + x - Ce^x

y(0) = 0: \quad y(x) = 1 + x - e^x
Slope fields

\[ y' = y - x \]

\[ y(x) = 1 + x - Ce^x \]

\[ y(0) = 0 : \quad y(x) = 1 + x - e^x \]

\[ y(0) = 1 : \quad y(x) = 1 + x \]
\[ y' = y - x \]

\[ y(x) = 1 + x - Ce^x \]

\[ y(0) = 0 : \ y(x) = 1 + x - e^x \]

\[ y(0) = 1 : \ y(x) = 1 + x \]

\[ y(0) = 2 : \ y(x) = 1 + x + e^x \]

Slopefield of \( y' = y - x \)
1. Sketch the slope field for the following differential equations:

   (a) \( y' = \frac{y}{x} \)

   (b) \( y' = x - y \)

   (c) \( y' = y^3 - y \)

2. Assignment: IMM1 - Tutorial 1.1
Euler’s method

\[ y' = f(x, y) \]

- Recall that a derivative is the limit of a difference quotient

\[
\frac{dy}{dx} = \lim_{h \to 0} \frac{y(x + h) - y(x)}{h}
\]
Euler’s method

\[ y' = f(x, y) \]

- Recall that a derivative is the limit of a difference quotient
  \[ \frac{dy}{dx} = \lim_{h \to 0} \frac{y(x + h) - y(x)}{h} \]

- For small \( h \) we have
  \[ \frac{y(x + h) - y(x)}{h} \approx y'(x) = f(x, y(x)), \]
  hence
  \[ y(x + h) \approx y(x) + hf(x, y(x)). \]
Euler’s method

The equation of the tangent line $\ell$ is

$$y = y_0 + (x - x_0)f(x_0, y_0).$$

Approximate $f(x_0 + h)$ with $y_0 + hf(x_0, y_0)$. 

$$y(x) = y(0) + \int_0^x f(\tau, y(\tau)) \, d\tau.$$
The equation of tangent line $\ell$ is $y = y_0 + (x - x_0)f(x_0, y_0)$. 
Euler’s method

The equation of tangent line $\ell$ is 
$$y = y_0 + (x - x_0)f(x_0, y_0).$$

Approximate $f(x_0 + h)$ with $y_0 + hf(x_0, y_0)$. 
Euler’s method

\[
\begin{aligned}
    y' &= f(x, y) \\
y(x_0) &= y_0
\end{aligned}
\]

- Fix the **step size** $h$.
- Make a table of points $(x_n, y_n)$, starting with $(x_0, y_0)$, where every point is calculated from the previous one with the equations

  \[
  \begin{aligned}
  x_{n+1} &= x_n + h \\
y_{n+1} &= y_n + hf(x_n, y_n)
  \end{aligned}
  \]
Euler’s method

\[
\frac{dy}{dx} = f(x, y) = y - x
\]

\[
y(0) = 0
\]

\[
y(x) = 1 + x - e^x
\]

Choose \( h = 0.5 \).

\[
\begin{array}{cccc}
n & x_n & y_n & y(x_n) \\ 0 & 0 & 0 & 0 \\ \end{array}
\]

Start in \((x_0, y_0) = (0, 0)\).
Euler’s method

\[
\frac{dy}{dx} = f(x, y) = y - x
\]

\[
y(0) = 0
\]

\[
y(x) = 1 + x - e^x
\]

Choose \( h = 0.5 \).

\[
\begin{array}{cccc}
    n & x_n & y_n & y(x_n) \\
    \hline
    0 & 0 & 0 & 0 \\
    1 & 0.5 & 0 & -0.149 \\
\end{array}
\]

The first Euler approximation is

\[
x_1 = x_0 + h = 0.5 \quad y_1 = y_0 + 0.5 \cdot (y_0 - 0) = 0
\]
Euler’s method

\[ \frac{dy}{dx} = f(x, y) = y - x \]

\[ y(0) = 0 \]

\[ y(x) = 1 + x - e^x \]

Choose \( h = 0.5 \).

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The second Euler approximation is

\[ x_2 = x_1 + h = 1 \]

\[ y_2 = y_1 + 0.5 \cdot (y_1 - 0.5) = -0.25 \]
Euler’s method

\[
\frac{dy}{dx} = f(x, y) = y - x
\]

\[
y(0) = 0
\]

\[
y(x) = 1 + x - e^x
\]

Choose \( h = 0.5 \).

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The third Euler approximation is

\[
x_3 = x_2 + h = 1.5 \quad y_3 = y_2 + 0.5 \cdot (y_2 - 1) = -0.875
\]
Euler’s method

\[ \frac{dy}{dx} = f(x, y) = y - x \]

\[ y(0) = 0 \]

\[ y(x) = 1 + x - e^x \]

Choose \( h = 0.5 \).

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The next Euler approximation is

\[ x_4 = x_3 + h = 2 \quad y_4 = y_3 + 0.5 \cdot (y_3 - 1.5) = -2.0625 \]
Euler’s method

\[
\frac{dy}{dx} = f(x, y) = y - x
\]

\[y(0) = 0\]

\[y(x) = 1 + x - e^x\]

Choose \(h = 0.5\).

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<tr>
<td>5</td>
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<td>-4.094</td>
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The last Euler approximation is

\[x_5 = x_4 + h = 2.5\] \[y_5 = y_4 + 0.5 \cdot (y_4 - 2) = -4.09375\]
Euler’s methods

Approximation becomes better by choosing smaller values for $h$. 

$h = \frac{1}{2}$

$h = \frac{1}{4}$

$h = \frac{1}{8}$
Assignment: IMM1 - Tutorial 1.2
Definition

A linear first order differential equation is a differential equation of the form

\[ y' + P(x)y = Q(x) \]

where \( P \) and \( Q \) are functions of \( x \).
**Definition**

A linear first order differential equation is a differential equation of the form

\[ y' + P(x)y = Q(x) \]

where \( P \) and \( Q \) are functions of \( x \).

- Notice that \( y' = f(x, y) \) with \( f(x, y) = Q(x) - P(x)y \).
A linear first order differential equation is a differential equation of the form

\[ y' + P(x)y = Q(x) \]

where \( P \) and \( Q \) are functions of \( x \).

- Notice that \( y' = f(x, y) \) with \( f(x, y) = Q(x) - P(x)y \).
- The equation is called first-order because it only contains the first derivative of \( y \).
Definition

A linear first order differential equation is a differential equation of the form

\[ y' + P(x)y = Q(x) \]

where \( P \) and \( Q \) are functions of \( x \).

- Notice that \( y' = f(x, y) \) with \( f(x, y) = Q(x) - P(x)y \).
- The equation is called first-order because it only contains the first derivative of \( y \).
- The equation is called linear because there are no nonlinear terms containing \( y \) and \( y' \), such as \( y^2 \) or \( \cos(y') \).
Assume \( v(x) \) is a function that satisfies the equation
\[
v' = Pv. \tag{1}
\]
Linear first order differential equations

\[
y' + Py = Q
\]

- Assume \( v(x) \) is a function that satisfies the equation
  \[
  v' = P v. 
  \]

- Then
  \[
  \frac{d}{dx}(vy) = vy' + v'y = vy' + vPy = v(y' + Py) = v Q(x). 
  \]
Linear first order differential equations

\[ y' + Py = Q \]

- Assume \( v(x) \) is a function that satisfies the equation
  \[ v' = Pv. \] (1)

- Then
  \[ \frac{d}{dx} (vy) = vy' + v'y = vy' + vPy = v(y' + Py) = v Q(x). \]

- Integrate left- and right-hand side
  \[ vy = \int v(x) Q(x) \, dx. \]
Assume $v(x)$ is a function that satisfies the equation

$$v' = Pv. \quad (1)$$

Then

$$\frac{d}{dx}(vy) = vy' + v'y = vy' + vPy = v(y' + Py) = vQ(x).$$

Integrate left- and right-hand side

$$vy = \int v(x)Q(x) \, dx.$$  

Divide left- and right-hand side by $v$:

$$y(x) = \frac{1}{v(x)} \int v(x)Q(x) \, dx.$$
Equation (1) is a **separable** differential equation that can be solved by integration (see lectures of week 2):

\[ v' = Pv \implies v(x) = e^{\int P(x) \, dx}, \]

where \( \int P(x) \, dx \) is an anti-derivative of \( P(x) \).

Solving a linear differential equation goes in two steps:

1. **Find the integrating factor** \( v \):
   
   \[ v(x) = e^{\int P(x) \, dx}. \]

2. **Find the solutions**:
   
   \[ y(x) = \frac{1}{v(x)} \int v(x) Q(x) \, dx. \]
The integrating factor and the complementary equation

\[ y' + P(x)y = Q(x) \]  \hspace{1cm} (1)

Let \( v(x) = e^{\int P(x) \, dx} \) be an integrating factor for the linear differential equation (1), then

\[ \frac{1}{v} = e^{-\int P(x) \, dx}, \]

and consequently

\[ \frac{d}{dx} \left( \frac{1}{v} \right) = \frac{d}{dx} e^{-\int P(x) \, dx} = -P(x)e^{-\int P(x) \, dx} = -P \frac{1}{v}. \]
The integrating factor and the complementary equation

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  \frac{d}{dx} \left( \frac{1}{v} \right) = \frac{d}{dx} e^{-\int P(x) \, dx} = -P(x) e^{-\int P(x) \, dx} = -P \frac{1}{v}.
  \]
- The function \( 1/v(x) \) is solution of the differential equation
  \[ y' + P(x)y = 0 \]  \hspace{1cm} (2)
The integrating factor and the complementary equation

\[ y' + P(x)y = Q(x) \] (1)

- Let \( v(x) = e^{\int P(x) \, dx} \) be an integrating factor for the linear differential equation (1), then

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- The function \( 1/v(x) \) is solution of the differential equation

\[ y' + P(x)y = 0 \] (2)

- Equation (2) is called the complementary equation of (1).
Always check your answer!

\[ y' + P(x)y = Q(x) \]  \hspace{1cm} (1)

For linear first-order differential equations, the solution is always of the form

\[ y(x) = \frac{1}{v(x)} \int v(x)Q(x) \, dx = g(x) + C h(x). \]
Always check your answer!

$$y' + P(x)y = Q(x)$$  \hspace{1cm} (1)

- For linear first-order differential equations, the solution is always of the form

$$y(x) = \frac{1}{v(x)} \int v(x) Q(x) \, dx = g(x) + C \, h(x).$$

### Check your answer

- The function $g(x)$ should satisfy equation (1).
- Function $h(x)$ should satisfy the complementary equation of (1).

- Note that $h(x) = \frac{1}{v(x)}$. 
The structure of the solutions of linear equations

\[ y' + P(x)y = Q(x) \]  

The solution of (1) is

\[ y(x) = \frac{1}{v} \int v(x) Q(x) \, dx. \]
The structure of the solutions of linear equations

The solution of (1) is

\[ y(x) = \frac{1}{v} \int v(x) Q(x) \, dx. \]

Let \( w(x) \) be an anti-derivative of \( v(x) Q(x) \), then

\[ y(x) = \frac{1}{v} (w(x) + C) = \frac{w(x)}{v(x)} + C \frac{1}{v(x)} \]
The structure of the solutions of linear equations

\[ y' + P(x)y = Q(x) \]  

(1)

- The solution of (1) is

\[ y(x) = \frac{1}{v} \int v(x)Q(x) \, dx. \]

- Let \( w(x) \) be an anti-derivative of \( v(x)Q(x) \), then

\[ y(x) = \frac{1}{v} (w(x) + C') = \frac{w(x)}{v(x)} + C \frac{1}{v(x)} \]

- The term \( w(x)/v(x) \) is called a **particular solution**.
The structure of the solutions of linear equations

\[ y' + P(x)y = Q(x) \]  

(1)

- The solution of (1) is
  \[ y(x) = \frac{1}{v} \int v(x)Q(x) \, dx. \]

- Let \( w(x) \) be an anti-derivative of \( v(x)Q(x) \), then
  \[ y(x) = \frac{1}{v}(w(x) + C) = \frac{w(x)}{v(x)} + C \frac{1}{v(x)} \]

- The term \( w(x)/v(x) \) is called a **particular solution**.
- The term \( C/v(x) \) is the general solution of the complementary equation of (1).
Example 1

\[ y' - 2x y = x \]

- \[ P(x) = -2x \] and \[ Q(x) = x. \]
Example 1

\[ y' - 2x y = x \]

- \( P(x) = -2x \) and \( Q(x) = x \).

- \( \int P(x) \, dx = \int -2x \, dx = -x^2 \), hence \( v(x) = e^{-x^2} \).

\[
\int P(x) \, dx = \int -2x \, dx = -x^2, \text{ hence } v(x) = e^{-x^2}.
\]
Example 1

\[
y' - 2xy = x
\]

- \( P(x) = -2x \) and \( Q(x) = x \).
- \( \int P(x) \, dx = \int -2x \, dx = -x^2 \), hence \( v(x) = e^{-x^2} \).
- Integrate \( v(x)Q(x) \):
  \[
  \int xe^{-x^2} \, dx = -\frac{1}{2} e^{-x^2} + C.
  \]
Example 1

\[ y' - 2xy = x \]

- \( P(x) = -2x \) and \( Q(x) = x \).
- \( \int P(x) \, dx = \int -2x \, dx = -x^2 \), hence \( v(x) = e^{-x^2} \).
- Integrate \( v(x)Q(x) \):
  \[ \int xe^{-x^2} \, dx = -\frac{1}{2} e^{-x^2} + C. \]
- Find \( y \):
  \[ y(x) = \frac{1}{v(x)} \int v(x)Q(x) \, dx = \frac{1}{e^{-x^2}} \left( -\frac{1}{2} e^{-x^2} + C \right) \]
  \[ = Ce^{x^2} - \frac{1}{2}. \]
Example 2

\[ xy' + 2y = x^3 \quad (x > 0) \]

- Rewrite the equation in the form \( y' + Py = Q \):

\[ y' + \frac{2}{x}y = x^2 \]
Example 2

\[ xy' + 2y = x^3 \quad (x > 0) \]

- Rewrite the equation in the form \( y' + Py = Q \):

\[
y' + \frac{2}{x}y = x^2 \quad \implies \quad P(x) = \frac{2}{x} \quad \text{and} \quad Q(x) = x^2 \quad (1)
\]
Example 2

\[
xy' + 2y = x^3 \quad (x > 0)
\]

- Rewrite the equation in the form \( y' + Py = Q \): \[
y' + \frac{2}{x} y = x^2 \quad \implies \quad P(x) = \frac{2}{x} \quad \text{and} \quad Q(x) = x^2 \tag{1}
\]
- Calculate the integrating factor:
  \[
  \int P(x) \, dx = \int \frac{2}{x} \, dx = 2 \ln(x) = \ln(x^2),
  \]
  \[
  v(x) = e^{\ln(x^2)} = x^2.
  \]
Example 2

\[ xy' + 2y = x^3 \quad (x > 0) \]

- Rewrite the equation in the form \( y' + Py = Q \):
  \[
y' + \frac{2}{x} y = x^2 \quad \Rightarrow \quad P(x) = \frac{2}{x} \quad \text{and} \quad Q(x) = x^2 \quad (1)
  \]

- Calculate the integrating factor:
  \[
  \int P(x) \, dx = \int \frac{2}{x} \, dx = 2 \ln(x) = \ln(x^2),
  \]
  \[
  v(x) = e^{\ln(x^2)} = x^2.
  \]

- Find \( y \):
  \[
  y(x) = \frac{1}{v(x)} \int v(x) Q(x) \, dx = \frac{1}{x^2} \int x^2 \cdot x^2 \, dx
  \]
  \[
  = \frac{1}{x^2} \int x^4 \, dx = \frac{1}{x^2} \left( \frac{1}{5} x^5 + C \right) = \frac{1}{5} x^3 + \frac{C}{x^2}.
  \]
Example 3

\[ y' + 2y = 2xe^{-x} \]

- \[ \int P(x) \, dx = \int 2 \, dx = 2x, \text{ hence } v(x) = e^{2x}. \]
Example 3

\[ y' + 2y = 2xe^{-x} \]

\[ \int P(x) \, dx = \int 2 \, dx = 2x, \text{ hence } v(x) = e^{2x}. \]

\[ y(x) = \frac{1}{v(x)} \int v(x)Q(x) \, dx \]

\[ = \frac{1}{e^{2x}} \int e^{2x} \cdot 2xe^{-x} \, dx = \frac{2}{e^{2x}} \int xe^x \, dx \]

\[ = 2e^{-2x}(xe^x - e^x + C) \]

\[ = 2(x - 1)e^{-x} + Ce^{-2x}. \]
\[
\begin{aligned}
\left\{
\begin{array}{l}
y' + 2y = 2xe^{-x} \\
y(0) = 1
\end{array}
\right.
\end{aligned}
\]
The general solution of the differential equation is
\[ y(x) = 2(x - 1)e^{-x} + Ce^{-2x}. \]
Example 4

\[
\begin{aligned}
\begin{cases}
y' + 2y = 2xe^{-x} \\
y(0) = 1
\end{cases}
\end{aligned}
\]

The general solution of the differential equation is

\[y(x) = 2(x - 1)e^{-x} + Ce^{-2x}.\]

From \(y(0) = 1\) follows

\[1 = y(0) = 2(0 - 1)e^{0} + Ce^{-2\cdot0} = -2 + C,
\]

hence \(C = 3.\)
The general solution of the differential equation is
\[ y(x) = 2(x - 1)e^{-x} + Ce^{-2x}. \]

From \( y(0) = 1 \) follows
\[ 1 = y(0) = 2(0 - 1)e^{0} + Ce^{-2\cdot0} = -2 + C, \]

hence \( C = 3. \)

The solution of (*) is
\[ y(x) = 2(x - 1)e^{-x} + 3e^{-2x}. \]
Assignment: IMM1 - Tutorial 1.3

Please fill out the questionnaire before you make the exercises
Assignment: IMM1 - Tutorial 1.3
**Inductor** — stores energy in a magnetic field

**Resistor** — limits the flow of current
**Ohm’s law for RL circuits**

\[ L \frac{di}{dt} + Ri(t) = V(t) \]
4.2

- Ohm’s law for RL circuits

\[ L \frac{di}{dt} + R \, i(t) = V(t) \]

- If we apply a constant voltage \( V(t) = V \) and close the circuit at \( t = 0 \) what will happen with the current \( i(t) \)?
\[ \left\{ \begin{array}{l}
\frac{d}{dt}i(t) + \frac{R}{L}i(t) = \frac{V}{L}, \\
i(0) = 0.
\end{array} \right. \]

\[ \int P(t) \, dt = \int \frac{R}{L} \, dt = \frac{Rt}{L}, \text{ hence } v(t) = e^{Rt/L}. \]
\[
\begin{aligned}
&\frac{d i}{dt} + \frac{R}{L} i(t) = \frac{V}{L}, \\
i(0) = 0.
\end{aligned}
\]

\[\int P(t) \, dt = \int \frac{R}{L} \, dt = \frac{Rt}{L}, \text{ hence } v(t) = e^{Rt/L}.\]

Find the general solution:
\[
i(t) = \frac{1}{e^{Rt/L}} \int \frac{V}{L} e^{Rt/L} \, dt = e^{-Rt/L} \left( \frac{V}{R} e^{Rt/L} + C \right)
\]
\[
= \frac{V}{R} + Ce^{-Rt/L}.
\]
Section 9.2, example 4

\[\begin{cases} 
\frac{di}{dt} + \frac{R}{L}i(t) = \frac{V}{L}, \\
i(0) = 0.
\end{cases}\]

- \[\int P(t) \, dt = \int \frac{R}{L} \, dt = \frac{Rt}{L}, \text{ hence } v(t) = e^{Rt/L}.\]

- Find the general solution:
  \[i(t) = \frac{1}{e^{Rt/L}} \int \frac{V}{L} e^{Rt/L} \, dt = e^{-Rt/L} \left( \frac{V}{R} e^{Rt/L} + C \right)\]
  \[= \frac{V}{R} + Ce^{-Rt/L}.\]

- Setting \(i(0) = 0\) we get \(C = -\frac{V}{R}\), hence
  \[i(t) = \frac{V}{R} \left( 1 - e^{-Rt/L} \right).\]
The differential equation

\[
\begin{aligned}
\frac{di}{dt} + \frac{R}{L}i(t) &= \frac{V}{L}, \\
i(0) &= 0.
\end{aligned}
\]

has the solution

\[
i(t) = \frac{V}{R} \left( 1 - e^{-Rt/L} \right).
\]
The differential equation

\[
\begin{cases}
\frac{di}{dt} + \frac{R}{L}i(t) = \frac{V}{L}, \\
i(0) = 0.
\end{cases}
\]

has the solution

\[i(t) = \frac{V}{R} \left(1 - e^{-\frac{Rt}{L}}\right).\]

The current will reach a steady state value

\[i_s = \lim_{t \to \infty} i(t) = \frac{V}{R}.\]
RL circuits

The current in an RL circuit is given by:

\[ i(t) = \frac{V}{R} e^{-\frac{t}{L/R}} \]

At time \( t = \frac{L}{R} \), about 63\% of the steady state current is reached.

At time \( t = 3 \frac{L}{R} \), about 95\% of the steady state current is reached.

The steady state is reached faster for smaller values of \( \frac{L}{R} \).
Step response: it takes time to reach the steady state current \( i_s = \frac{V}{R} \).
- Step response: it takes time to reach the steady state current $i_S = \frac{V}{R}$.
- At $t = \frac{L}{R}$ the current is $(1 - \frac{1}{e})i_S \approx 0.631i_S$. 
RL circuits

Step response: it takes time to reach the steady state current \( i_S = V/R \).

- At \( t = L/R \) the current is \( (1 - \frac{1}{e})i_S \approx 0.631i_S \).
- At \( t = 3L/R \) about 95% of the steady state current is reached.
- Step response: it takes time to reach the steady state current $i_S = V/R$.
- At $t = L/R$ the current is $(1 - 1/e)i_S \approx 0.631i_S$.
- At $t = 3L/R$ about 95% of the steady state current is reached.
- The steady state is reached faster for smaller values of $L/R$. 
Low-pass filters

Subwoofer

Equalizer
Consider a circuit with $R = L = 1$ (to simplify the algebra) and an oscillating voltage source $V(t) = \cos(\omega t)$.

$$\frac{di}{dt} + i(t) = \cos(\omega t)$$
Consider a circuit with $R = L = 1$ (to simplify the algebra) and an oscillating voltage source $V(t) = \cos(\omega t)$.

\[
\frac{d}{dt} i(t) + i(t) = \cos(\omega t)
\]

We will show that after a while the solution is

\[
i(t) = \frac{1}{\sqrt{1 + \omega^2}} \cos(\omega t - \varphi)
\]

where $\varphi$ is a \textbf{phase shift} that depends on the frequency $\omega$. 
Consider a circuit with $R = L = 1$ (to simplify the algebra) and an oscillating voltage source $V(t) = \cos(\omega t)$.

\[
\frac{di}{dt} + i(t) = \cos(\omega t)
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We will show that after a while the solution is

\[
i(t) = \frac{1}{\sqrt{1 + \omega^2}} \cos(\omega t - \varphi)
\]

where $\varphi$ is a \textbf{phase shift} that depends on the frequency $\omega$.

With “after a while” we mean that $i(t) \approx \frac{1}{\sqrt{1 + \omega^2}} \cos(\omega t - \varphi)$ for large values of $t$. 
\[
\frac{d}{dt}i + i(t) = \cos(\omega t)
\]

\[
\int P(t) \, dt = \int 1 \, dt = t, \text{ hence } v(t) = e^t.
\]
Low-pass filters

\[ \frac{d i}{d t} + i(t) = \cos(\omega t) \]

\[ \int P(t) \, dt = \int 1 \, dt = t, \text{ hence } v(t) = e^t. \]
\[ \frac{d i}{d t} + i(t) = \cos(\omega t) \]

\[ \int P(t) \, dt = \int 1 \, dt = t, \text{ hence } v(t) = e^t. \]

Find the general solution:

\[ i(t) = \frac{1}{v(t)} \int v(t) \cos(\omega t) \, dt \]

\[ = e^{-t} \int e^t \cos(\omega t) \, dt. \]
Use integration by parts twice:

\[
\int e^t \cos(\omega t) \, dt = e^t \cos(\omega t) - \int e^t \cdot -\omega \sin(\omega t) \, dt
\]
Use integration by parts twice:

\[
\int e^t \cos(\omega t) \, dt = e^t \cos(\omega t) - \int e^t \cdot -\omega \sin(\omega t) \, dt
\]
Use integration by parts twice:

\[\int e^t \cos(\omega t) \, dt = e^t \cos(\omega t) - \int e^t \cdot -\omega \sin(\omega t) \, dt\]

\[= e^t \cos(\omega t) + \omega \int e^t \sin(\omega t) \, dt\]
Use integration by parts twice:

\[
\int e^t \cos(\omega t) \, dt = e^t \cos(\omega t) - \int e^t \cdot -\omega \sin(\omega t) \, dt
\]

\[
= e^t \cos(\omega t) + \omega \int e^t \sin(\omega t) \, dt
\]

\[
= e^t \cos(\omega t) + \omega \left( e^t \sin(\omega t) - \int e^t \cdot \omega \cos(\omega t) \, dt \right)
\]
Use integration by parts twice:

\[
\int e^t \cos(\omega t) \, dt = e^t \cos(\omega t) - \int e^t \cdot -\omega \sin(\omega t) \, dt \\
= e^t \cos(\omega t) + \omega \int e^t \sin(\omega t) \, dt \\
= e^t \cos(\omega t) + \omega \left( e^t \sin(\omega t) - \int e^t \cdot \omega \cos(\omega t) \, dt \right) \\
= e^t \cos(\omega t) + \omega e^t \sin(\omega t) - \omega^2 \int e^t \cos(\omega t) \, dt.
\]

This gives

\[
\int e^t \cos(\omega t) \, dt = e^t \left[ \frac{1}{1 + \omega^2} \cos(\omega t) + \frac{\omega}{1 + \omega^2} \sin(\omega t) \right] + C.
\]
The general solution is

\[ i(t) = e^{-t} \int e^t \cos(\omega t) \, dt = \frac{1}{1 + \omega^2} \cos(\omega t) + \frac{\omega}{1 + \omega^2} \sin(\omega t) + Ce^{-t}. \]
Low-pass filters

\[ \frac{di}{dt} + i(t) = \cos(\omega t) \]

- The general solution is

\[ i(t) = e^{-t} \int e^{t} \cos(\omega t) \, dt \]

\[ = \frac{1}{1 + \omega^2} \cos(\omega t) + \frac{\omega}{1 + \omega^2} \sin(\omega t) + Ce^{-t}. \]

- For large values of \( t \) the term \( Ce^{-t} \) is small, so we may neglect this term:

\[ i(t) = \frac{1}{1 + \omega^2} \cos(\omega t) + \frac{\omega}{1 + \omega^2} \sin(\omega t). \]
\[
\cos(\varphi) = \frac{1}{\sqrt{1 + \omega^2}} \quad \text{and} \quad \sin(\varphi) = \frac{\omega}{\sqrt{1 + \omega^2}}
\]
Low-pass filters

\[ \frac{d i}{d t} + i(t) = \cos(\omega t) \]

The solution is

\[ i(t) = \frac{1}{1 + \omega^2} \cos(\omega t) + \frac{\omega}{1 + \omega^2} \sin(\omega t) \]
Low-pass filters

\[
\frac{di}{dt} + i(t) = \cos(\omega t)
\]

The solution is

\[
i(t) = \frac{1}{1 + \omega^2} \cos(\omega t) + \frac{\omega}{1 + \omega^2} \sin(\omega t)
\]

\[
= \frac{1}{\sqrt{1 + \omega^2}} \left[ \frac{1}{\sqrt{1 + \omega^2}} \cos(\omega t) + \frac{\omega}{\sqrt{1 + \omega^2}} \sin(\omega t) \right]
\]
\[ \frac{d i}{dt} + i(t) = \cos(\omega t) \]

The solution is

\[ i(t) = \frac{1}{1 + \omega^2} \cos(\omega t) + \frac{\omega}{1 + \omega^2} \sin(\omega t) \]

\[ = \frac{1}{\sqrt{1 + \omega^2}} \left[ \frac{1}{\sqrt{1 + \omega^2}} \cos(\omega t) + \frac{\omega}{\sqrt{1 + \omega^2}} \sin(\omega t) \right] \]

\[ = \frac{1}{\sqrt{1 + \omega^2}} \left( \cos(\varphi) \cos(\omega t) + \sin(\varphi) \sin(\omega t) \right) \]
Low-pass filters

\[ \frac{d}{dt} i(t) + i(t) = \cos(\omega t) \]

The solution is

\[ i(t) = \frac{1}{1 + \omega^2} \cos(\omega t) + \frac{\omega}{1 + \omega^2} \sin(\omega t) \]

\[ = \frac{1}{\sqrt{1 + \omega^2}} \left[ \frac{1}{\sqrt{1 + \omega^2}} \cos(\omega t) + \frac{\omega}{\sqrt{1 + \omega^2}} \sin(\omega t) \right] \]

\[ = \frac{1}{\sqrt{1 + \omega^2}} \left( \cos(\phi) \cos(\omega t) + \sin(\phi) \sin(\omega t) \right) \]

\[ = \frac{1}{\sqrt{1 + \omega^2}} \cos(\omega t - \phi). \]
The solution is

\[ i(t) = \frac{1}{\sqrt{1 + \omega^2}} \cos(\omega t - \varphi). \]

Conclusions:

- For a sinusoidal input the output is a shifted cosine.
- The output’s amplitude depends on the input’s frequency.
- High frequencies are damped stronger than low frequencies.
Low-pass filters

\[ L \frac{d i}{d t} + R i(t) = \cos(\omega t) \]

The general solution is

\[ i(t) = \frac{1}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t - \varphi) + Ce^{\frac{-Rt}{L}}. \]
The general solution is
\[ i(t) = \frac{1}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t - \varphi) + Ce^{-Rt/L}. \]

If we neglect the exponential term then
\[ i(t) = \frac{1}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t - \varphi). \]
Amplitude as a function of frequency (linear plot).

The parameters $R$ and $L$ determine the cut-off frequency.
Low-pass filters

Amplitude as a function of frequency (log-log plot).

The parameters $R$ and $L$ determine the cut-off frequency.
The voltage over the capacitor $C$ is $u(t)$. The function $u(t)$ satisfies the differential equation

$$RC \frac{du}{dt} + u = V(t).$$

See also: *Introduction to Physical Systems.*

Find $u(t)$ for the following voltages $V(t)$ and initial conditions:

(a) $V(t) = 1$ and $u(0) = 0$.  
(b) $V(t) = 0$ and $u(0) = 1$.  
(c) $V(t) = e^{-t}$ and $u(0) = 0$.  
(d) $V(t) = \sin t$ and $u(0) = 0$. 